Lecture 13-14

Introduction to High-Resolution Schemes

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Godunov's Method

1D conservation law in PDE form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

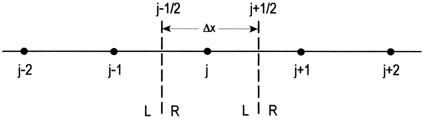
Integral form in space

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u(x,t)\mathrm{d}x = -\{f[u(b,t)] - f[u(a,t)]\}$$

Integral form in time and space

$$\int_a^b u(x,t_{n+1})\mathrm{d}x - \int_a^b u(x,t_n)\mathrm{d}x = -\Delta t\{\bar{f}[u(b,t)] - \bar{f}[u(a,t)]\}$$

Godunov's Method



Control volume in one dimension

Godunov's Method

$$\int_a^b u(x, t_{n+1})\mathrm{d}x - \int_a^b u(x, t_n)\mathrm{d}x = -\Delta t\{\bar{f}[u(b,t)] - \bar{f}[u(a,t)]\}$$

Introduce the cell average as the unknown

$$\overline{u_j^n} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\mathrm{d}x$$

Gives

$$\overline{u_j^{n+1}} - \overline{u_j^n} = -\frac{\Delta t}{\Delta x}\{\bar{f}[u(x_{j+1/2}, t)] - \bar{f}[u(x_{j-1/2}, t)]\}$$

Godunov's Method

Piecewise constant reconstruction

$$u(x, t_n) = \overline{u_j^n} \qquad x_{j-1/2} \leq x \leq x_{j+1/2}$$

Gives the following left and right states at the $j + 1/2$ interface:

$$u^{\mathrm{L}}_{j+1/2} = \bar{u}_j \qquad \text{and} \qquad u^{\mathrm{R}}_{j+1/2} = \bar{u}_{j+1}$$

Define

$$u^* \left(\frac{x}{t}, u^{\mathrm{L}}, u^{\mathrm{R}} \right)$$

as the exact solution to the local Riemann problem given by

$$\begin{aligned} u &= u^{\mathrm{L}} \qquad x < 0 \\ u &= u^{\mathrm{R}} \qquad x \geq 0 \end{aligned}$$

Godunov's Method

One Riemann problem centered at $x_{j+1/2}, t_n$

$$u(x,t) = u^* \left(\frac{x - x_{j+1/2}}{t - t_n}, \overline{u_j^n}, \overline{u_{j+1}^n} \right)$$

Another centered at $x_{j-1/2}, t_n$

$$u(x,t) = u^* \left(\frac{x - x_{j-1/2}}{t - t_n}, \overline{u_{j-1}^n}, \overline{u_j^n} \right)$$

Time step restriction to prevent interacting Reimann problems:

$$|a_{\max}| \Delta t < \frac{\Delta x}{2}$$

Godunov's Method

Advance in time by including the contribution from the Riemann problem centered at $x_{j-1/2}$ and that centered at $x_{j+1/2}$ and integrating to get the cell average:

$$
\begin{aligned}
\overline{u_j^{n+1}} &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})\mathrm{d}x \\
&= \frac{1}{\Delta x}\left[\int_{x_{j-1/2}}^{x_j} u^*\left(\frac{x - x_{j-1/2}}{\Delta t}, \overline{u_{j-1}^n}, \overline{u_j^n}\right)\mathrm{d}x\right. \\
&\qquad \left. + \int_{x_j}^{x_{j+1/2}} u^*\left(\frac{x - x_{j+1/2}}{\Delta t}, \overline{u_j^n}, \overline{u_{j+1}^n}\right)\mathrm{d}x\right]
\end{aligned}
$$

Godunov's Method

Consider the linear convection equation with positive wave speed

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Exact solution provides solution to Riemann problem:

$$u(x,t) = u_n\left(x - a(t - t_n)\right)$$

No left-moving waves, so time step restriction becomes

$$a\Delta t \leq \Delta x$$

Godunov's Method

Piecewise constant reconstruction

$$\text{cell } j-1 \qquad u(x,t_n) = \overline{u_{j-1}^n}$$

$$\text{cell } j \qquad u(x,t_n) = \overline{u_j^n}$$

Advance in time using the two solutions to the Riemann problems

$$\begin{aligned}
\overline{u_j^{n+1}} &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_{n+1})\mathrm{d}x \\
&= \frac{1}{\Delta x}\left[a\Delta t\overline{u_{j-1}^n} + (\Delta x - a\Delta t)\overline{u_j^n}\right] \\
&= \overline{u_j^n} - \frac{a\Delta t}{\Delta x}(\overline{u_j^n} - \overline{u_{j-1}^n})
\end{aligned}$$

Equivalent to first-order backward in space, explicit Euler time marching

Godunov's Method

Riemann solution is constant along rays (constant x/t)

Therefore the flux at $x = 0$ is constant in time

We can remove the time average on the flux to obtain:

$$\overline{u_j^{n+1}} = \overline{u_j^n} - \frac{\Delta t}{\Delta x}\left[f(u^*(0, \overline{u_j^n}, \overline{u_{j+1}^n})) - f(u^*(0, \overline{u_{j-1}^n}, \overline{u_j^n}))\right]$$

Godunov's method can therefore be written in the generic form:

$$\overline{u_j^{n+1}} = \overline{u_j^n} - \frac{\Delta t}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2})$$

with the numerical flux function

$$\hat{f}_{j+1/2} = f(u^*(0, \overline{u_j^n}, \overline{u_{j+1}^n}))$$

Godunov's Method

Semi-discrete form:

$$\frac{\mathrm{d}\overline{u_j}}{\mathrm{d}t} = -\frac{1}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2})$$

Godunov's numerical flux function for the linear convection equation

$$\hat{f}_{j+1/2} = \frac{1}{2}(a + |a|)\overline{u_j^n} + \frac{1}{2}(a - |a|)\overline{u_{j+1}^n}$$

Gives upwind flux for both positive and negative a

Godunov's Method

Nonlinear example: Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

$$\hat{f}_{j+1/2} = \begin{cases} \frac{1}{2}u_{j+1}^2 & \text{if} \quad u_j,\ u_{j+1} \text{ are both } \leq 0 \\ \frac{1}{2}u_j^2 & \text{if} \quad u_j,\ u_{j+1} \text{ are both } \geq 0 \\ 0 & \text{if} \quad u_j \leq 0 \leq u_{j+1} \\ \frac{1}{2}u_j^2 & \text{if} \quad u_j > 0 \geq u_{j+1} \text{ and } |u_j| \geq |u_{j+1}| \\ \frac{1}{2}u_{j+1}^2 & \text{if} \quad u_j \geq 0 > u_{j+1} \text{ and } |u_j| \leq |u_{j+1}| \end{cases}$$

Roe's Approximate Riemann Solver

Roe's Approximate Riemann Solver

Quasi-linear form of the 1D Euler equations:

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = 0$$

Locally linearized form:

$$\frac{\partial Q}{\partial t} + \bar{A}\frac{\partial Q}{\partial x} = 0$$

Exact solution to the locally linearized can be determined from the eigensystem of $\bar{A}$

Roe's Approximate Riemann Solver

Roe chose the average state to satisfy the following:

$$f^R - f^L = A(\bar{Q})(Q^R - Q^L)$$

For the Euler equations this gives

$$\begin{aligned}
\bar{\rho} &= \sqrt{\rho^L \rho^R} \\
\bar{u} &= \frac{(u\sqrt{\rho})^L + (u\sqrt{\rho})^R}{\sqrt{\rho^L} + \sqrt{\rho^R}} \\
\bar{H} &= \frac{(H\sqrt{\rho})^L + (H\sqrt{\rho})^R}{\sqrt{\rho^L} + \sqrt{\rho^R}}
\end{aligned}$$

Roe's Approximate Riemann Solver

Decoupled form:

$$\frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial x} = 0$$

Recouple to get

$$\begin{aligned}
\hat{f}_{j+1/2} &= X\left[\frac{1}{2}(\Lambda + |\Lambda|)W_j + \frac{1}{2}(\Lambda - |\Lambda|)W_{j+1}\right] \\
&= X\left[\frac{1}{2}(\Lambda + |\Lambda|)X^{-1}Q_j + \frac{1}{2}(\Lambda - |\Lambda|)X^{-1}Q_{j+1}\right] \\
&= \frac{1}{2}X\Lambda X^{-1}(Q_j + Q_{j+1}) + \frac{1}{2}X|\Lambda|X^{-1}(Q_j - Q_{j+1}) \\
&= \frac{1}{2}\bar{A}(Q_j + Q_{j+1}) + \frac{1}{2}|\bar{A}|(Q_j - Q_{j+1})
\end{aligned}$$

Roe's Approximate Riemann Solver

Replace the first term with a standard centered flux to get the numerical flux function for Roe's method

$$\hat{f}_{j+1/2} \quad = \quad \frac{1}{2}(f_j + f_{j+1}) + \frac{1}{2}|\bar{A}|(Q_j - Q_{j+1})$$

In the scalar case, the Roe numerical flux function becomes

$$\hat{f}_{j+1/2} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|\bar{a}_{j+1/2}|(u_{j+1} - u_j)$$

with

$$\bar{a}_{j+1/2} = \begin{cases} \frac{f_{j+1} - f_j}{u_{j+1} - u_j} & \text{if } u_{j+1} \neq u_j \\ \\ a(u_j) & \text{if } u_{j+1} = u_j \end{cases}$$

Roe's Approximate Riemann Solver

For Burgers equation ($f = u^2/2$) we get

$$\bar{a}_{j+1/2} = \frac{\frac{u_{j+1}^2}{2} - \frac{u_j^2}{2}}{u_{j+1} - u_j} = \frac{1}{2}(u_j + u_{j+1})$$

$$\begin{aligned} \hat{f}_{j+1/2} &= \frac{1}{2}(\frac{1}{2}u_j^2 + \frac{1}{2}u_{j+1}^2) - \frac{1}{2}|\frac{1}{2}(u_j + u_{j+1})|(u_{j+1} - u_j) \\ &= \begin{cases} \frac{1}{2}u_{j+1}^2 & \text{if } \bar{a}_{j+1/2} \le 0 \\ \frac{1}{2}u_j^2 & \text{if } \bar{a}_{j+1/2} > 0 \end{cases} \end{aligned}$$

Differs from Godunov when $u_j \le 0 \le u_{j+1}$ (Godunov's flux equals 0)

Roe's Approximate Riemann Solver

As a result, the Roe scheme can permit expansion shocks

This can be corrected by a simple "entropy fix"

One approach involves replacing the eigenvalues $\lambda = u + a$ and $\lambda = u - a$ by

$$\frac{1}{2}\left(\frac{\lambda^2}{\epsilon} + \epsilon\right)$$

if they are less than or equal to ϵ, where ϵ is a small parameter

Higher-Order Reconstruction

High-Order Reconstruction

Thus far we have based our numerical flux at the interface on the states u_j and u_{j+1}

$$\hat{f}_{j+1/2} = \hat{f}(u_j, u_{j+1})$$

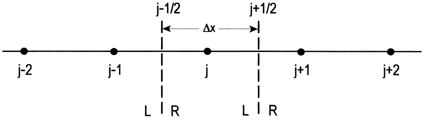
These are the left and right states at the interface if piecewise constant reconstruction is used, leading to schemes that are first order in space

For higher order we generalize to

$$\hat{f}_{j+1/2} = \hat{f}(u^L_{j+1/2}, u^R_{j+1/2})$$

where $u^L_{j+1/2}$ and $u^R_{j+1/2}$ are determined from the reconstruction in cells j and $j+1$, respectively

High-Order Reconstruction



Control volume in one dimension

High-Order Reconstruction

The following gives a piecewise constant reconstruction if $\alpha = \beta = 0$, piecwise linear if $\alpha = 1$, $\beta = 0$, piecewise quadratic if $\alpha = \beta = 1$:

$$
\begin{aligned}
u(x) \quad = \quad & \overline{u_j} + \alpha \left(\frac{\overline{u_{j+1}} - \overline{u_{j-1}}}{2\Delta x} \right) (x - x_j) \\
& + \beta \left(\frac{\overline{u_{j+1}} - 2\overline{u_j} + \overline{u_{j-1}}}{2\Delta x^2} \right) [(x - x_j)^2 - \frac{\Delta x^2}{12}]
\end{aligned}
$$

High-Order Reconstruction

Substitute $x = x_j + \Delta x/2$ into the reconstruction in cell j to get

$$u^L_{j+1/2} = u_j + \frac{1}{4}[(\alpha - \beta/3)(u_j - u_{j-1}) + (\alpha + \beta/3)(u_{j+1} - u_j)]$$

Substitute $x = x_{j+1} - \Delta x/2$ into the reconstruction in cell $j+1$ to get

$$u^R_{j+1/2} = u_{j+1} - \frac{1}{4}[(\alpha + \beta/3)(u_{j+1} - u_j) + (\alpha - \beta/3)(u_{j+2} - u_{j+1})]$$

Substitute into

$$\hat{f}_{j+1/2} = \hat{f}(u^L_{j+1/2}, u^R_{j+1/2})$$

and finally into

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{1}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2})$$

High-Order Reconstruction

Linear convection equation, positive a, upwind flux function $\hat{f}_{j+1/2} = a u^L_{j+1/2}$

Linear reconstruction $\alpha = 1$ gives:

$$u^L_{j+1/2} = u_j + \frac{1}{4}(u_{j+1} - u_{j-1})$$

Semi-discrete form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_j = -\frac{a}{4\Delta x}(u_{j+1} + 3u_j - 5u_{j-1} + u_{j-2})$$

Second-order antisymmetric part, third-order symmetric (dissipative) part

High-Order Reconstruction

Linear convection equation, positive a, upwind flux function $\hat{f}_{j+1/2} = a u^L_{j+1/2}$

Quadratic reconstruction $\alpha = \beta = 1$ gives:

$$u^L_{j+1/2} = \frac{1}{6}(2u_{j+1} + 5u_j - u_{j-1})$$

Semi-discrete form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_j = -\frac{a}{6\Delta x}(2u_{j+1} + 3u_j - 6u_{j-1} + u_{j-2})$$

Fourth-order antisymmetric part, third-order symmetric (dissipative) part

Conservation Laws and Total Variation

Conservation Laws and Total Variation

1D scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

Exact solution is constant along characteristics unless the characteristics intersect to form a shock wave:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(u) = \frac{\partial f}{\partial u}$$

For an initial value problem, the total variation between any pairs of characteristics is conserved, where the total variation is defined as

$$TV(u(x,t)) = \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x} \right| \mathrm{d}x$$

Conservation Laws and Total Variation

In the presence of discontinuities, the total variation is nonincreasing in time if the discontinuities satisfy an entropy inequality:

$$TV(u(x, t_0 + t)) \leq TV(u(x, t_0))$$

As a consequence, local maxima do not increase, local minima do not decrease, and monotonic solutions remain monotonic, i.e. no new extrema are created

Ideally numerical schemes would ensure that the numerical solution retains these properties

Monotone and Monotonicity-Preserving Schemes

Monotone and Monotonicity-Preserving Schemes

Consider a conservative discretization of a conservation law written as follows:

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}) \\ &= H(u_{j-l}^n, u_{j-l+1}^n, \ldots, u_{j+l}^n) \end{aligned}$$

where $$\hat{f}_{j+1/2} = \hat{f}(u_{j-l+1}, \ldots, u_{j+l})$$

The method is *monotone* if H is a monotone increasing function of each of its arguments:

$$\frac{\partial H}{\partial u_i}(u_{-l}, \ldots, u_{+l}) \geq 0 \quad \text{for all} \quad -l \leq i \leq l$$

Limited to first-order accuracy

Monotone and Monotonicity-Preserving Schemes

Example: linear convection equation with positive a discretized with first-order backward differencing in space and explicit Euler time marching:

$$u_j^{n+1} = C_\mathrm{n} u_{j-1}^n + (1 - C_\mathrm{n}) u_j^n$$

where

$$C_\mathrm{n} = \frac{a \Delta t}{\Delta x}$$

$$\frac{\partial H}{\partial u_{j-1}} = C_\mathrm{n} \qquad \frac{\partial H}{\partial u_j} = 1 - C_\mathrm{n}$$

Monotone when stable ($C_\mathrm{n} \leq 1$)

Monotone and Monotonicity-Preserving Schemes

Lax-Wendroff method (2nd-order in space and time)

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{1}{2}\frac{ah}{\Delta x}(u_{j+1}^n - u_{j-1}^n) \\ &\quad + \frac{1}{2}\left(\frac{ah}{\Delta x}\right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \end{aligned}$$

$$u_j^{n+1} = \frac{C_\mathrm{n}}{2}(1 + C_\mathrm{n})u_{j-1}^n + (1 - C_\mathrm{n}^2)u_j^n + \frac{C_\mathrm{n}}{2}(C_\mathrm{n} - 1)u_{j+1}^n$$

Monotone condition violated even for $C_\mathrm{n} \leq 1$

Monotone and Monotonicity-Preserving Schemes

Weaker condition: A scheme is monotonicity preserving if monotonicity of u^n guarantees monotonicity of u^{n+1}, where a solution is monotonic if

$$\min(u_{j-1}, u_{j+1}) \leq u_j \leq \max(u_{j-1}, u_{j+1}) \quad \text{for all } j$$

The monotonicity preserving property is sufficient to ensure that no new extrema are created, local maxima are nonincreasing, and local minima are nondecreasing

All monotone schemes are monotonicity preserving but not vice versa

All linear monotonicity preserving schemes are at most first order

Higher than first order monotonicity preserving schemes must be nonlinear

Total-Variation-Diminishing Conditions

Total-Variation-Diminishing Conditions

What about a total variation diminishing (TVD) property as a design condition for a numerical scheme?

Discrete total variation:

$$TV_{\mathrm{d}}(u) = \sum_{-\infty}^{\infty} |u_j - u_{j-1}|$$

All monotone schemes are TVD

All TVD schemes are monotonicity preserving

To be higher than first order, TVD schemes must be nonlinear

Total-Variation-Diminishing Conditions

Rewrite the semi-discrete form of a conservative scheme

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{1}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2})$$

as

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = \frac{1}{\Delta x}[C^-_{j+1/2}(u_{j+1} - u_j) - C^+_{j-1/2}(u_j - u_{j-1})]$$

The TVD conditions are:

$$C^-_{j+1/2} \geq 0 \quad \text{and} \quad C^+_{j-1/2} \geq 0$$

Total-Variation-Diminishing Conditions

For the fully discrete form

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}[C^-_{j+1/2}(u^n_{j+1} - u^n_j) - C^+_{j-1/2}(u^n_j - u^n_{j-1})]$$

we have the following additional condition:

$$1 - \frac{\Delta t}{\Delta x}(C^-_{j+1/2} + C^+_{j-1/2}) \geq 0$$

Total-Variation-Diminishing Conditions

Examples: linear convection equation with positive a

First-order backward in space, explicit Euler in time:

$$
\begin{aligned}
u_j^{n+1} &= u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) \\
&= u_j^n + \frac{\Delta t}{\Delta x}[0 \cdot (u_{j+1} - u_j) - a(u_j - u_{j-1})]
\end{aligned}
$$

First two conditions are met

Third condition coincides with the requirement for stability

If $C_{\mathrm{n}} \leq 1$, the scheme is TVD

Total-Variation-Diminishing Conditions

What about second-order backward in space?

Semi-discrete form:

$$\begin{aligned}
\frac{\mathrm{d}u_j}{\mathrm{d}t} &= -\frac{a}{2\Delta x}(3u_j - 4u_{j-1} + u_{j-2}) \\
&= \frac{1}{\Delta x}\left[0 \cdot (u_{j+1} - u_j) - \frac{a}{2}(3(u_j - u_{j-1}) - (u_{j-1} - u_{j-2}))\right] \\
&= \frac{1}{\Delta x}\left[0 \cdot (u_{j+1} - u_j) - \frac{a}{2}\left(3 - \frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}}\right)(u_j - u_{j-1})\right]
\end{aligned}$$

$$C^{-}_{j+1/2} = 0\ , \qquad C^{+}_{j-1/2} = \frac{a}{2}\left(3 - \frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}}\right)$$

Not TVD

Total-Variation-Diminishing Conditions

Violates TVD condition if

$$\frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}} > 3$$

Note that if we fit a parabola to three points it is monotonic if

$$\frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}} \leq 3$$

and nonmonotonic if

$$\frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}} > 3$$

Total-Variation-Diminishing Schemes with Limiters

Total-Variation-Diminishing Schemes with Limiters

Write the 2nd-order backward scheme as a 1st-order scheme with a correction for 2nd-order accuracy:

$$
\begin{aligned}
\frac{\mathrm{d}u_j}{\mathrm{d}t} &= -\frac{a}{2\Delta x}(3u_j - 4u_{j-1} + u_{j-2}) \\
&= -\frac{a}{\Delta x}[(u_j - u_{j-1}) + \underbrace{\frac{1}{2}(u_j - u_{j-1}) - \frac{1}{2}(u_{j-1} - u_{j-2})]}_{\text{for second-order accuracy}}
\end{aligned}
$$

This can be written in conservative form with the flux function

$$\hat{f}_{j+1/2} = \frac{a}{2}(3u_j - u_{j-1})$$

or equivalently $\quad \hat{f}_{j+1/2} = a[u_j + \underbrace{\frac{1}{2}(u_j - u_{j-1})}_{\text{for second-order}}]$

Total-Variation-Diminishing Schemes with Limiters

Now let's limit the second-order correction

$$\hat{f}_{j+1/2} = a[u_j + \frac{1}{2}\psi_j(u_j - u_{j-1})]$$

where $\psi = 0$ gives the 1st-order method, and $\psi = 1$ gives the 2nd-order method

to obtain

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{a}{\Delta x}[(u_j - u_{j-1}) + \frac{1}{2}\psi_j(u_j - u_{j-1}) - \frac{1}{2}\psi_{j-1}(u_{j-1} - u_{j-2})]$$

Total-Variation-Diminishing Schemes with Limiters

Recall that whether or not the TVD condition is violated depends on the ratio

$$\frac{u_{j-1} - u_{j-2}}{u_j - u_{j-1}}$$

Hence define the ratio

$$r_j = \frac{u_{j+1} - u_j}{u_j - u_{j-1}}$$

and the limiter function

$$\psi_j = \psi(r_j) \geq 0$$

Total-Variation-Diminishing Schemes with Limiters

Substituting in we obtain

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\underbrace{a\left[1 + \frac{1}{2}\psi(r_j) - \frac{1}{2}\frac{\psi(r_{j-1})}{r_{j-1}}\right]}_{C^+_{j-1/2}}(u_j - u_{j-1})$$

To satisfy the TVD condition we require

$$\frac{\psi(r_{j-1})}{r_{j-1}} - \psi(r_j) \leq 2$$

Since $\psi(r_j) \geq 0$, this is satisfied if

$$\psi(r_{j-1}) \leq 2r_{j-1}$$

Total-Variation-Diminishing Schemes with Limiters

Symmetry requires that

$$\psi\left(\tfrac{1}{r}\right) \quad = \frac{\psi(r)}{r}$$

from which it follows that

$$\psi(r) \quad \leq 2$$

Therefore we have the following conditions on the limiter function $\psi(r)$

$$\begin{aligned}
\psi(r) &\geq 0 \quad \text{for} \quad r \geq 0 \\
\psi(r) &= 0 \quad \text{for} \quad r \leq 0 \\
\psi(r) &\leq 2r \\
\psi\left(r\right) &\leq 2 \\
\psi(\tfrac{1}{r}) &= \frac{\psi(r)}{r}
\end{aligned}$$

We also want $\psi(1) = 1$ for second-order accuracy

Total-Variation-Diminishing Schemes with Limiters

Minmod

$$\psi = \begin{cases} \min(r, 1) & r > 0 \\ \\ 0 & r \leq 0 \end{cases}$$

Superbee

$$\psi(r) = \max[0,\ \min(2r, 1),\ \min(r, 2)]$$

van Leer

$$\psi(r) = \frac{r + |r|}{1 + r}$$

Total-Variation-Diminishing Schemes with Limiters

van Albada

$$\psi = \begin{cases} \frac{r^2+r}{1+r^2} & r > 0 \\ 0 & r \leq 0 \end{cases}$$

Total-Variation-Diminishing Schemes with Limiters

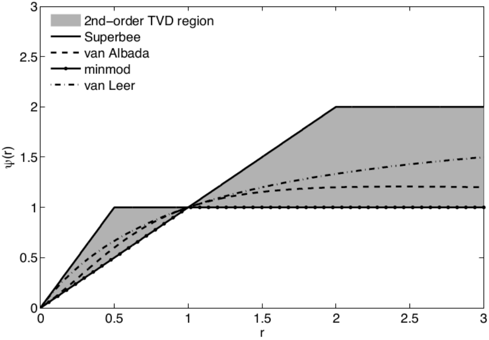


Diagram showing the second-order TVD region for limiters and several well-known limiter functions

Total-Variation-Diminishing Schemes with Limiters

Applying flux limiters to a split scheme (scalar):

$$f^+ = \frac{1}{2}(a + |a|)u \qquad f^- = \frac{1}{2}(a - |a|)u$$

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = -\frac{1}{\Delta x}[(f_j^+ - f_{j-1}^+) + \frac{1}{2}\psi(r_j)(f_j^+ - f_{j-1}^+) - \frac{1}{2}\psi(r_{j-1})(f_{j-1}^+ - f_{j-2}^+)]$$

$$-\frac{1}{\Delta x}[(f_{j+1}^- - f_j^-) + \frac{1}{2}\psi\left(\frac{1}{r_j}\right)(f_{j+1}^- - f_j^-) - \frac{1}{2}\psi\left(\frac{1}{r_{j+1}}\right)(f_{j+2}^- - f_{j+1}^-)]$$

Total-Variation-Diminishing Schemes with Limiters

Consider applying a limiter to the slope of a linear reconstruction

$$\hat{f}_{j+1/2} = au^L_{j+1/2} = au_j + \frac{a}{4}\phi(r_j)(u_{j+1} - u_{j-1})$$

Determine $\phi(r_j)$ such that this is equivalent to the previous scheme:

$$\begin{aligned}
\hat{f}_{j+1/2} &= a[u_j + \frac{1}{2}\psi(r_j)(u_j - u_{j-1})] \\
&= a\left[u_j + \frac{1}{4}\psi(r_j)\frac{2(u_j - u_{j-1})}{(u_{j+1} - u_{j-1})}(u_{j+1} - u_{j-1})\right] \\
&= a\left[u_j + \frac{1}{4}\psi(r_j)\frac{2}{r_j + 1}(u_{j+1} - u_{j-1})\right]
\end{aligned}$$

Total-Variation-Diminishing Schemes with Limiters

Gives

$$\phi(r) = \frac{2}{r+1}\psi(r)$$

Easy to implement in finite-volume solvers

Total-Variation-Diminishing Schemes with Limiters

Can write $\phi(r)$ in terms of

$$\Delta_+ = u_{j+1} - u_j \qquad \Delta_- = u_j - u_{j-1}$$

For example, for $r > 0$ minmod can be written as

$$\phi_j = \frac{2}{\Delta_+ + \Delta_-} \min(\Delta_+, \Delta_-)$$

Hence minmod replaces the slope

$$\frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

by the lesser of $$\frac{u_{j+1} - u_j}{\Delta x} \quad \text{and} \quad \frac{u_j - u_{j-1}}{\Delta x}$$

Total-Variation-Diminishing Schemes with Limiters

Similarly for $r > 0$ the van Leer limiter can be written as

$$\phi_j = \frac{4\Delta_+\Delta_-}{(\Delta_+ + \Delta_-)^2}$$

So the slope of the linear reconstruction becomes

$$\frac{1}{\Delta x}\left(\frac{2\Delta_+\Delta_-}{\Delta_+ + \Delta_-}\right)$$

As r tends to infinity ($\Delta^+ \gg \Delta^-$) the slope tends to

$$\frac{2(u_j - u_{j-1})}{\Delta x}$$

Total-Variation-Diminishing Schemes with Limiters

As r tends to zero ($\Delta^- \gg \Delta^+$) the slope tends to

$$\frac{2(u_{j+1} - u_j)}{\Delta x}$$

Hence in these limits the van Leer limiter produces a slope that is twice that resulting from the minmod limiter

Total-Variation-Diminishing Schemes with Limiters

- limiter chatter
- systems of equations
- multiple dimensions
- time marching methods